

Bordism Field Theories II

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Abstract

This paper introduces interstices on warped product manifolds, which act as discrete regulators on an otherwise smooth spacetime. We analyze flux quantization and compactification in the presence of interstitial points, and prove a theorem about a lemma about manifolds arising from fibered products.

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Disclaimer

An appendix on errata has been added. We thank Richard Pinçak for helpful critique and discussion.

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1 Interstitial Fields

Let $\mathcal{M} = \mathcal{M}^{(9|2)}$ be an eleven-dimensional (pseudo-)Riemannian supermanifold with or without corners and intrinsic curvature. Suppose we have a family of fields

$$\text{Fam}_{\mathcal{M}} = \bigoplus_i^{f(m_j+n_k)} \Phi_{i \in \mathcal{I}}^{(m_j|n_k)} \cdot (m_j + n_k \leq 11)$$

varying smoothly over $\mathcal{M}$, where $\mathcal{I} \simeq \mathbb{Q}$ is a rational subring of $\mathbb{Q}$, and $f(m_j + n_k) > 0$ is a function of the sum of odd and even dimensions. From here on out, we will let $\lceil m_j \rceil = \hat{m}$ and $\lceil n_k \rceil = \hat{n}$. Following [1], we have two copies of a sphere; however, rather than $S^2 \times S^2$, we get $S^{\hat{m}-2}$ and $S^{\hat{n}-2}$, and for every i , we have a bundle

$$\text{Bun}_{\Phi_i^{(m_j|n_k)}} : \mathcal{M} \longrightarrow (S^{\hat{m}-2} \times S^{\hat{n}-2})^{\text{codim } S^{\min(\hat{m}, \hat{n}-2)}}$$

and the *generic fiber* of this bundle winds around the first sphere $\hat{m}$ times and the second $\hat{n}$ times. Again, this extends to a family of its own:

$$\text{Fam}_{\text{Bun}_{\Phi_i^{(m_j|n_k)}}} : \text{Fam}_{\mathcal{M}} \longrightarrow \bigoplus_{\hat{m}+\hat{n}}^{f(9+2)} (S^{\hat{m}-2} \times S^{\hat{n}-2})^{\text{codim } S^{\min(\hat{m}, \hat{n}-2)}}$$

where again the function is strictly increasing.

Notation 1.1. We shall use $\widetilde{\times}$ to denote a fibered product; i.e., $\mathbf{A} \widetilde{\times} \mathbf{B} = \mathbf{A} \times_{\sqsupset} \mathbf{B}$, where $\sqsupset : \text{Fib}(\mathbf{A}, \mathbf{B})$ is the fiber and $\text{Fib}(\mathbf{A}, \mathbf{B})$ is the category of fibrations $\pi(\mathbf{A}) \longrightarrow \pi(\mathbf{B})$.

This notation is introduced, because $S^{\hat{m}-2} \widetilde{\times} S^{\hat{n}-2}$ is actually an *Einstein* or *warped product* manifold is chosen with respect to a *special point* $\wp$ lying in $(T_x(\mathbf{A}) \cap T_y^*(\mathbf{B})) \cup (T_{x'}^*(\mathbf{A}) \cup T_{y'}(\mathbf{B}))$, which we shall call the *interstice* of $\mathbf{A}$ and $\mathbf{B}$. We call the special point an *interstitial point* if an action is conserved with respect to $\wp$, or colloquially, if there is a local mirror symmetry around the special point.

Definition 1.1. The interstice of two groupoids $\mathbf{A}$ and $\mathbf{B}$ is the product

$$\mathcal{T}\mathbf{A} \times_{\text{Fib}(\mathbf{A}, \mathbf{B})} \mathcal{T}^*\mathbf{B}$$

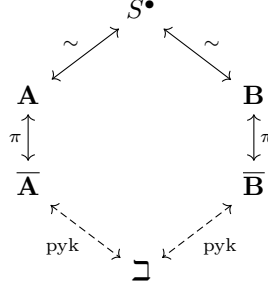
where $\widetilde{\times}$ is a 2-categorical product.

Notation 1.2. We shall denote the interstice of two groupoids by $\text{inter}_{\mathbf{A}, \mathbf{B}}$.

Lemma 1.1. If $\mathbf{A} \cong \mathbf{B}$ via weak equivalence, then $\pi(\text{Fib}(\mathbf{A}, \mathbf{B}))$ is étale.

Proof. First, it must be shown that $\pi(\text{Fib}(\mathbf{A}, \mathbf{B}))$ is étale. First, note that due to the space-groupoid equivalence, $\mathbf{A}$ and $\mathbf{B}$ are both groupoids. Since the weak equivalence is assumed, we have $\mathbf{B} = f([0, 1]) \times \mathbf{A}$. Since the unit interval is contractable, then any homeomorphism must also be so. Thus, the contractability of $\mathbf{B}$ depends on that of $\mathbf{A}$; i.e., if $\mathcal{T}\mathbf{A}$ is smooth with respect to infinitesimal deformation, then $\mathcal{T}^*\mathbf{B}$ will also be smooth, and thus so too will the interstice be.

We obtain the following diagram:



and since $\square \ni \varphi$ is pyknotic, it condenses to φ in the Zariski topology. Notice that every morphism in the above diagram is invertible; therefore, the category forms a one-object subcategory of $\mathcal{Gpd}\mathcal{E}t$, and each of the objects are 1-small subobjects. $\square$

Remark 1.1. See Appendix A for more details on this proof.

1.1 BFT

1.1.1 Bulk BFT

In BF field theories, the *integration measure*, which is typically given by a $E = d^d x$ term, is replaced by the action:

$$S_{BF} = \text{Tr}(B \wedge F) \quad (1.1)$$

where B is a $(d-2)$ -form valued in the adjoint representation of a gauge group G ,

$$F = dA + A \wedge A$$

is the curvature 2-form on a gauge field A , and the integral is taken over a generic d -dimensional spacetime.

When we superize this theory¹, instead of the normal $d^d x$ term, we make the replacement

$$E = d^{\hat{m}} x_\mu + d^{\hat{n}} x_\nu$$

where the left-hand term is the *even action* and the right-hand term is the *odd action*, and the odd action acts a Lagrangian multiplier or “boost” to the even one. However, if we have

$$d^{\hat{m}} x_\mu + d^{\hat{n}} x_\nu = 0$$

we are left with an anomaly.

If $\square_{/\mathcal{C}}$ is a context² over a category $\square$ and $s \in \square \simeq \hat{b} \in \mathbb{L}^{\hat{m}|\hat{n}}$ is an object in $\square$, then the anomaly means that for a measurement $q(E)$ of E , we get

$$q(E) = \nu_{\mathbb{E}} \iff d^{\hat{m}} x_\mu + d^{\hat{n}} x_\nu = 0$$

¹I.e., equip it with a $\mathbb{Z}_2$ grade

²See [3] for a refresher

where $\nu_{\mathbb{E}} \sim \emptyset$ is Emmerson's element³. Thus, the anomalous wave will *not* be observed or detected by any measuring device.

To cancel the anomaly, first consider a 2-categorical BF theory, where rather than merely having a gauge field alongside a Lagrangian multiplier, we promote these to *higher categorical* objects. We extend the BF action to interstices:

$$S_{\text{int}} = \int_{\mathcal{M}} \text{Tr}(B \wedge F) + \int_{\Sigma=\text{inter}_{\mathbf{A},\mathbf{B}}} \text{Tr}(C \wedge R) \quad (1.2)$$

where:

- B is a $(d-2)$ -form, as in ordinary bulk BFT
- C is a new $(d-3)$ -form field supported *only* on the interstice

and

- R is the curvature of the connection in a higher groupoid, encoding how flux propagates between the two regions

and R describes how the gauge field A “jumps” across Σ .

The action of a bulk BFT is topological, and possesses a gauge invariance:

$$A \longrightarrow A + d\lambda \quad B \longrightarrow B + d\Lambda$$

where λ is a local gauge transformation parameter governing the infinitesimal deformation of individual groupoid fibers and Λ is a higher order one, controlling the deformation of the entire fiber structure in a manner compatible with étale descent. They satisfy an inverse limit condition:

$$\Lambda = \varprojlim \lambda_i$$

where the λ_i are local gauge parameters for the individual étale covers indexed by i , and Λ is the total gauge freedom emerging from condensation.

1.2 Interstice as a Defect Theory

The interstice is a codimension-1 hypersurface Σ ; it can serve as a boundary condition of sorts, and it is embedded as:

$$\Sigma = M_+ \sqcup M_-$$

where M_+ and M_- are two adjacent regions. The field C , which is defined locally on Σ mediate between these two regions, acting as a sort of *regulator* which controls the resolution of observation.

As the action S_{int} varies, so too should there be a corresponding variation in the anomaly inflow equation:

$$S_{\text{inflow}} = \frac{1}{2} (q(E) + q(Fam_{\mathcal{M}})) \quad (1.3)$$

³See [4] and [5]

Rather than the usual gauge group G , we consider 2-groups or *Lie groupoids* R , replacing A by $A + \Gamma$ where Γ is a 1-form connection on a 2-group bundle. We define R by:

$$R = F_+ - F_-$$

Let's assume we have a principle G -bundle over the bulk M with a gauge connection A , leading to the field strength:

$$F = dA + A \wedge A$$

On the interstice, the gauge field is *not necessarily* smooth, leading to a jump condition:

$$F_+ - F_- = dR + A \wedge R + R \wedge A$$

Under a gauge transformation

$$A \longrightarrow A + d\lambda + [A, \lambda]$$

, the interstitial field C transforms as

$$R \longrightarrow R + [R, \Lambda]$$

To directly compute the flux, we use the equation:

$$\Phi = \int_{\Sigma} R$$

which we are safe to re-write as

$$\Phi = \int_{\Sigma} (F_+ - F_-)$$

and if $G = dC + C \wedge C$, then by Stoke's theorem we get:

$$\Phi = \int_{\partial\Sigma} C$$

2 Refining the Freund-Rubin Ansatz for Bordism Field Theories

The usual Freund-Rubin compactification [1] assumes the following ansatz for an 11-dimensional SUGRA metric:

$$ds^2 = ds_{\text{AdS}_d}^2 + L^2 ds_X^2$$

where:

- AdS_d is a d -dimensional anti-de Sitter space
- X is a compact internal space, traditionally assumed to be $S^{k \leq 9}$

and

- L is the characteristic length scale determined by the fluxes

Here, we shall modify the Freund-Rubin four-form field strength, F_4 , which is

$$F_{\mu\nu\rho\sigma} \sim \Lambda \epsilon_{\mu\nu\rho\sigma}$$

with Λ proportional to the vacuum energy and $\epsilon_{\mu\nu\rho\sigma}$ the volume form of AdS_d , while the rest of the components of F_4 vanish except possibly along the compact space X . Following in the spirit of [2], and in this document, we define the following *warping function*:

$$f(m_j + n_k) = \int_{\text{Fam}_\Sigma} \omega_{\mathcal{M}, \mathcal{N}} \quad (2.1)$$

where Fam_Σ is the interstitial bordism family over compact spaces, and $\omega_{\mathcal{M}, \mathcal{N}}$ is a differential form encoding the topological transition data between two regions.

2.1 Flux Quantization

Since F_4 satisfies

$$dF_4 = 0 \quad d \star F_4 \neq 0$$

the flux is constrained by the topology. However, in the presence of interstices, the jump condition

$$F_4^{(\text{Bulk})} - F_4^{(\text{Interstitial})} = dC$$

which suggests that the bordism family structure modifies flux quantization via:

$$\int_{\text{Fam}_\mathcal{M}} F_4 = \sum_i \int_{X_i} F_{4,i} \neq 0 \quad (2.2)$$

Furthermore, the four-form traditionally satisfies [11]:

$$\int_{X_4} F_4 \in 2\pi\mathbb{Z} \quad (2.3)$$

which becomes

$$\int_{X_4} F_4 \in 2\pi(\mathcal{I} \simeq \mathbb{Q}) \quad (2.4)$$

The flux is no longer a single conserved quantity, but is distributed across a family of compactifications:

$$\text{Fam}_{F_4} = \bigoplus_i^{f(m_j+n_k)} F_{4,i}^{(m_j|n_k)}$$

For a given bordism family $\text{Fam}_\mathcal{M}$, flux quantization must hold on the *entire family*, leading to:

$$\sum_i \int_{X_{4,i}} F_4 = \int_{\text{Fam}_\mathcal{M}} F_4 \in 2\pi\mathbb{Q} \quad (2.5)$$

and, by letting $X_{4,+}$ denote the compact space on one side of Σ and $X_{4,-}$ the right-hand compact space, we get

$$\int_{X_{4,+}} F_{4,+} - \int_{X_{4,-}} F_{4,-} = \int_\Sigma dC \quad (2.6)$$

This suggests that flux can discontinuously jump across the interstice, leading to a generalized conservation law where dC represents anomaly inflow at the interstice.

3 Bordism Correction Terms

The bordism group Ω_n^U contains torsion components, meaning that certain bordism classes do not bound a manifold but contribute discrete factors. This implies downstream constraints in, e.g., flux quantization.

Consider the *long exact sequence* in homology associated with a bordism W :

$$\dots \rightarrow H_{n+1}(W) \rightarrow H_n(\partial W) \rightarrow H_n(W) \rightarrow H_{n-1}(\partial W) \rightarrow \dots$$

Whence H_n has torsion, then we must modify the flux quantization condition by appending:

$$\int_{\Upsilon_{\text{out},1}} F_4 - \int_{\Upsilon_{\text{out},2}} = \int_{\Sigma} dB + \text{Tor}(\Omega_*^U) \quad (3.1)$$

where each of the $\Upsilon_{\text{out},*}$ are outputs of the bordism, $W : \Upsilon_{\text{in},0} \rightrightarrows \Upsilon_{\text{out},1} \times \Upsilon_{\text{out},2}$ is the bordism itself, and for every submanifold $\Upsilon_{\bullet,*}$, there is an associated Hilbert space $\mathcal{H}_{\Upsilon_{\bullet,*}}$ so that the bordism is a bifurcation of a unitary state living on $\mathcal{H}_{\Upsilon_{\text{in},0}}$. The output manifolds represent possible states⁴, which may be considered as entangled or in superposition.

Remark 3.1. *The Hilbert space splits as a pair of maps context slices over a spacetime groupoid:*

$$\mathcal{M}_{/\square} \xrightleftharpoons[g]{f} \mathcal{M}_{/\diamond_1} \times \mathcal{M}_{/\diamond_2}$$

where the collection of interstitial loci⁵ $[\wp] : \square$ may or may not be preserved, depending upon the severity of dissipation across time slices. We could, in principle, extend this to arbitrarily many future states by constructing a family

$$\text{Fam}_{\mathcal{M}_{/\diamond_{>0}}} = \text{Hom}(\mathcal{M}_{/\square}, \mathcal{M}_{/\diamond_{>0}})$$

and by letting

$$\square x = \diamond^0 x = \diamond_0$$

for all “packets” $x \in \mathcal{M}$.

To superize, we let $f \sim g$ for every function $(g \neq f) \in \text{Hom}(\mathcal{M}_{/\square}, \mathcal{M}_{/\diamond_{>0}})$ and take

$$\deg[f] = (-1)^{\deg \square + \deg \diamond_{>0}}$$

where $\deg$ is the Fredholm index. Then, if x is a quantum, its image under any $\tilde{f} \in [f]$ is a “squantum,” or superpartner.

3.1 Explicit Computations

The complex bordism groups Ω_n^U are given by the homotopy groups of the *Thom spectrum* MU :

$$\Omega_*^U = \pi_* MU$$

⁴Or, using the language of modal logic, $\diamond$ states

⁵Also known as “collar germs”

For low dimensions, the groups contain torsion elements; in particular:

$$\Omega_2^U = 0 \quad \Omega_4^U = \mathbb{Z} \quad \Omega_6^U = 0 \quad \Omega_8^U = \mathbb{Z}/2$$

The $\mathbb{Z}/2$ factor in Ω_8^U means that there exists an 8-manifold $\mathcal{M}^8$ which represents a non-trivial bordism class of order 2; that is, this manifold bounds a complex 9-manifold twice. Thus, we modify (3.1) appropriately:

$$\int_{\Sigma_{\text{interstice}}} dB \equiv 0 \pmod{2} \quad (3.2)$$

A well-known generator of the torsion in Ω_8^U is the complex projective space $\mathbb{C}P^4$ with a *specific framing*. The fundamental fact is that $2\mathbb{C}P^4$ is bordant to zero in Ω_8^U , meaning that twice this manifold bounds a 9-dimensional complex-oriented manifold.

Thus, we take

$$\mathcal{M}^8 \cong \mathbb{C}P^4$$

which carries a natural complex structure, with tangent bundle related to tautological line bundle $\mathcal{O}(-1)$ over $\mathbb{C}P^4$.

Since

$$H^4(\mathbb{C}P^4) = \mathbb{Z}$$

a flux F_4 on this space satisfies the quantization condition

$$\int_{\mathbb{C}P^4} F_4 \in \mathbb{Z}$$

However, if $\mathbb{C}P^4$ represents a torsion class in bordism, then twice this flux must trivialize in some 9-dimensional bounding space, which introduces a mod 2 quantization condition. So, we kick the can down the road a bit further, and modify (3.2) to become

$$\int_{\Upsilon_{\text{out},1}} F_4 - \int_{\Upsilon_{\text{out},2}} F_4 = \int_{\Sigma} dB + k \pmod{2} \quad (3.3)$$

thus constraining the spectrum of allowed flux values on the interstice.

3.2 Modified Partition Function

A quantum path integral must sum over all possible flux configuration, including the twisted sector. A good partition function would be:

$$\mathcal{Z} = \sum_{[F_4]} e^{iS[F_4]} \cdot e^{i\Theta([F_4])}$$

where $S[F_4]$ is the standard action term and $\Theta([F_4])$ is a torsion-dependent quantum phase factor that introduces interference between topologically distinct sectors.

A Pyknotic Objects

Recollection A.1. *A pyknotic structure means that objects carry a pro-finite topology, enabling a sort of inverse limit topology:*

$$\mathbf{A} \leftarrow \mathbf{A}_1 \leftarrow \dots \leftarrow \mathbf{A}_2 \dots$$

where each of the $\mathbf{A}_i$ are progressively higher resolution étale covers; $\mathbf{A}$ then emerges as the limiting object:

$$\mathbf{A} = \varprojlim \mathbf{A}_i$$

The limit $\varprojlim \mathbf{A}_i$ is naturally a stack over the étale site, which means that:

$$H_{\text{ét}}^n(\mathbf{A}) = \varprojlim H_{\text{ét}}^n(\mathbf{A}_i)$$

Remark A.1. *The above proof relies on the work of Barwick and Haine [8]. The condensation map $\mathbf{pyk}$ ensures that the category collapses to a smallest étale subcategory, effectively selecting the most refined structure. Since Zariski sheaves are already local in the étale topology, the condensation essentially restricts us to the “correct” structure.*

1-smallness follows from:

Proposition A.1. *The limit is well-defined in the étale topology, remains a 1-small subobject, and respects étale descent. Hence, condensation ensures that $\varprojlim \mathbf{A}_i$ stays fully faithful and remains in $\mathcal{Gpd}^{\text{ét}}$.*

Proof. For any two objects x and y in the inverse system, the mapping space:

$$\text{Hom}_{\varprojlim \mathbf{A}_i}(x, y)$$

may be computed as a limit of the corresponding Hom-sets at each stage:

$$\varprojlim \text{Hom}_{\mathbf{A}_i}(x_i, y_i)$$

Since each $\mathbf{A}_i$ is étale, its morphisms are locally homeomorphic to a space over a base. That is, for every object x_i , there is a small neighborhood U_i such that $\pi : U_i \rightarrow S^\bullet$ is a local diffeomorphism. This means every morphism lifts uniquely in the étale topology, and there are no extra degrees of freedom in the inverse system.

The condensation property of $\mathbf{\square}$ ensures that at the limit, the étale transition maps glue uniquely; that is, there are no trivial kernels in our inverse limit. This is because pyknosis enforces Zariski-local triviality of the interstitial morphisms, meaning that all the lifting maps remain étale.

ÉtGpd behaves as a single atomic object, and the proposition is shown to be true. $\square$

B Errata

B.1 Lemma 1.1

In the étale topology, smoothness must be established via local properties, such as the existence of atlases or effective descent in the context of Lie groupoids.

B.2 Higher Gauge Structures and Descent

Gauge transformations are usually local and glue over topological or étale covers via cocycle data.

Let $\{\lambda_i\}_{i \in I}$ be a collection of local gauge transformations defined over a cover $\{U_i\}$ of a base space X , with compatibility data λ_{ij} on overlaps $U_i \cap U_j$. A global gauge transformation Λ arises via descent if the stack of fields satisfies effective descent.

We avoid interpreting unless working in a category where pro-objects or formal completions are defined.

B.3 Bordism

The correct structure of the complex bordism group in dimension 8 is:

$$\Omega_8^U \cong \mathbb{Z} \oplus \mathbb{Z}/24$$

This replaces the earlier incorrect claim of $\mathbb{Z}/2\mathbb{Z}$. The presence of a free part and a 24-torsion component has implications for both cobordism classification and physical applications, such as anomaly cancellation and quantization.

Given the above correction, we replace

$$\int_{\Sigma} dB \cong 0 \pmod{2}$$

by

$$\int_{\Sigma} dB \cong 0 \pmod{24}$$

B.4 Relevance to SUGRA

Latly, we address concerns of relevance to 11-dimensional supergravity. The interstices are interpreted as codimension-1 topological defects or domain walls. Their interaction with the equations of motion in 11D supergravity must be made explicit.

The field equation is

$$d * G = \frac{1}{2} G \wedge G$$

and, to be physically consistent, the interstices should either

- source G , as in brane-like objects (e.g. $M9$ -branes), or
- mediate transitions between different flux vacua, like phase interfaces.

We propose modeling them via jump conditions or cobordism data between field configurations, though a detailed treatment is left for future work.

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D Disclaimer

Artificial Intelligence was used for the following purpose(s): Construction of Appendix A

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varying smoothly over $\mathcal{M}$, where $\mathcal{I} \simeq \mathbb{Q}$ is a rational subring of $\mathbb{Q}$, and $f(m_j + n_k) > 0$ is a function of the sum of odd and even dimensions. From here on out, we will let $\lceil m_j \rceil = \hat{m}$ and $\lceil n_k \rceil = \hat{n}$. Following [1], we have two copies of a sphere; however, rather than $S^2 \times S^2$, we get $S^{\hat{m}-2}$ and $S^{\hat{n}-2}$, and for every i , we have a bundle

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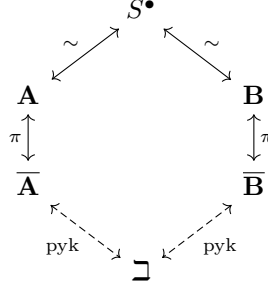
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Notation 1.2. We shall denote the *interstice* of two groupoids by $\text{inter}_{\mathbf{A}, \mathbf{B}}$.

Lemma 1.1. If $\mathbf{A} \cong \mathbf{B}$ via weak equivalence, then $\pi(\text{Fib}(\mathbf{A}, \mathbf{B}))$ is étale.

Proof. First, it must be shown that $\pi(\text{Fib}(\mathbf{A}, \mathbf{B}))$ is étale. First, note that due to the space-groupoid equivalence, $\mathbf{A}$ and $\mathbf{B}$ are both groupoids. Since the weak equivalence is assumed, we have $\mathbf{B} = f([0, 1]) \times \mathbf{A}$. Since the unit interval is contractable, then any homeomorphism must also be so. Thus, the contractability of $\mathbf{B}$ depends on that of $\mathbf{A}$; i.e., if $\mathcal{T}\mathbf{A}$ is smooth with respect to infinitesimal deformation, then $\mathcal{T}^*\mathbf{B}$ will also be smooth, and thus so too will the interstice be.

We obtain the following diagram:



and since $\square \ni \varphi$ is pyknotic, it condenses to φ in the Zariski topology. Notice that every morphism in the above diagram is invertible; therefore, the category forms a one-object subcategory of $\mathcal{Gpd}\mathcal{E}t$, and each of the objects are 1-small subobjects. $\square$

Remark 1.1. See Appendix A for more details on this proof.

1.1 BFT

1.1.1 Bulk BFT

In BF field theories, the *integration measure*, which is typically given by a $E = d^d x$ term, is replaced by the action:

$$S_{BF} = \text{Tr}(B \wedge F) \quad (1.1)$$

where B is a $(d-2)$ -form valued in the adjoint representation of a gauge group G ,

$$F = dA + A \wedge A$$

is the curvature 2-form on a gauge field A , and the integral is taken over a generic d -dimensional spacetime.

When we superize this theory¹, instead of the normal $d^d x$ term, we make the replacement

$$E = d^{\hat{m}} x_\mu + d^{\hat{n}} x_\nu$$

where the left-hand term is the *even action* and the right-hand term is the *odd action*, and the odd action acts a Lagrangian multiplier or “boost” to the even one. However, if we have

$$d^{\hat{m}} x_\mu + d^{\hat{n}} x_\nu = 0$$

we are left with an anomaly.

If $\square/\mathcal{C}$ is a context² over a category $\square$ and $s \in \square \simeq \hat{b} \in \mathbb{L}^{\hat{m}|\hat{n}}$ is an object in $\square$, then the anomaly means that for a measurement $q(E)$ of E , we get

$$q(E) = \nu_{\mathbb{E}} \iff d^{\hat{m}} x_\mu + d^{\hat{n}} x_\nu = 0$$

¹I.e., equip it with a $\mathbb{Z}_2$ grade

²See [3] for a refresher

where $\nu_{\mathbb{E}} \sim \emptyset$ is Emmerson's element³. Thus, the anomalous wave will *not* be observed or detected by any measuring device.

To cancel the anomaly, first consider a 2-categorical BF theory, where rather than merely having a gauge field alongside a Lagrangian multiplier, we promote these to *higher categorical* objects. We extend the BF action to interstices:

$$S_{\text{int}} = \int_{\mathcal{M}} \text{Tr}(B \wedge F) + \int_{\Sigma=\text{inter}_{\mathbf{A},\mathbf{B}}} \text{Tr}(C \wedge R) \quad (1.2)$$

where:

- B is a $(d-2)$ -form, as in ordinary bulk BFT
- C is a new $(d-3)$ -form field supported *only* on the interstice

and

- R is the curvature of the connection in a higher groupoid, encoding how flux propagates between the two regions

and R describes how the gauge field A “jumps” across Σ .

The action of a bulk BFT is topological, and possesses a gauge invariance:

$$A \longrightarrow A + d\lambda \quad B \longrightarrow B + d\Lambda$$

where λ is a local gauge transformation parameter governing the infinitesimal deformation of individual groupoid fibers and Λ is a higher order one, controlling the deformation of the entire fiber structure in a manner compatible with étale descent. They satisfy an inverse limit condition:

$$\Lambda = \varprojlim \lambda_i$$

where the λ_i are local gauge parameters for the individual étale covers indexed by i , and Λ is the total gauge freedom emerging from condensation.

1.2 Interstice as a Defect Theory

The interstice is a codimension-1 hypersurface Σ ; it can serve as a boundary condition of sorts, and it is embedded as:

$$\Sigma = M_+ \sqcup M_-$$

where M_+ and M_- are two adjacent regions. The field C , which is defined locally on Σ mediate between these two regions, acting as a sort of *regulator* which controls the resolution of observation.

As the action S_{int} varies, so too should there be a corresponding variation in the anomaly inflow equation:

$$S_{\text{inflow}} = \frac{1}{2} (q(E) + q(Fam_{\mathcal{M}})) \quad (1.3)$$

³See [4] and [5]

Rather than the usual gauge group G , we consider 2-groups or *Lie groupoids* R , replacing A by $A + \Gamma$ where Γ is a 1-form connection on a 2-group bundle. We define R by:

$$R = F_+ - F_-$$

Let's assume we have a principle G -bundle over the bulk M with a gauge connection A , leading to the field strength:

$$F = dA + A \wedge A$$

On the interstice, the gauge field is *not necessarily* smooth, leading to a jump condition:

$$F_+ - F_- = dR + A \wedge R + R \wedge A$$

Under a gauge transformation

$$A \longrightarrow A + d\lambda + [A, \lambda]$$

, the interstitial field C transforms as

$$R \longrightarrow R + [R, \Lambda]$$

To directly compute the flux, we use the equation:

$$\Phi = \int_{\Sigma} R$$

which we are safe to re-write as

$$\Phi = \int_{\Sigma} (F_+ - F_-)$$

and if $G = dC + C \wedge C$, then by Stoke's theorem we get:

$$\Phi = \int_{\partial\Sigma} C$$

2 Refining the Freund-Rubin Ansatz for Bordism Field Theories

The usual Freund-Rubin compactification [1] assumes the following ansatz for an 11-dimensional SUGRA metric:

$$ds^2 = ds_{\text{AdS}_d}^2 + L^2 ds_X^2$$

where:

- AdS_d is a d -dimensional anti-de Sitter space
- X is a compact internal space, traditionally assumed to be $S^{k \leq 9}$

and

- L is the characteristic length scale determined by the fluxes

Here, we shall modify the Freund-Rubin four-form field strength, F_4 , which is

$$F_{\mu\nu\rho\sigma} \sim \Lambda \epsilon_{\mu\nu\rho\sigma}$$

with Λ proportional to the vacuum energy and $\epsilon_{\mu\nu\rho\sigma}$ the volume form of AdS_d , while the rest of the components of F_4 vanish except possibly along the compact space X . Following in the spirit of [2], and in this document, we define the following *warping function*:

$$f(m_j + n_k) = \int_{\text{Fam}_\Sigma} \omega_{\mathcal{M}, \mathcal{N}} \quad (2.1)$$

where Fam_Σ is the interstitial bordism family over compact spaces, and $\omega_{\mathcal{M}, \mathcal{N}}$ is a differential form encoding the topological transition data between two regions.

2.1 Flux Quantization

Since F_4 satisfies

$$dF_4 = 0 \quad d \star F_4 \neq 0$$

the flux is constrained by the topology. However, in the presence of interstices, the jump condition

$$F_4^{(\text{Bulk})} - F_4^{(\text{Interstitial})} = dC$$

which suggests that the bordism family structure modifies flux quantization via:

$$\int_{\text{Fam}_\mathcal{M}} F_4 = \sum_i \int_{X_i} F_{4,i} \neq 0 \quad (2.2)$$

Furthermore, the four-form traditionally satisfies [11]:

$$\int_{X_4} F_4 \in 2\pi\mathbb{Z} \quad (2.3)$$

which becomes

$$\int_{X_4} F_4 \in 2\pi(\mathcal{I} \simeq \mathbb{Q}) \quad (2.4)$$

The flux is no longer a single conserved quantity, but is distributed across a family of compactifications:

$$\text{Fam}_{F_4} = \bigoplus_i^{f(m_j+n_k)} F_{4,i}^{(m_j|n_k)}$$

For a given bordism family $\text{Fam}_\mathcal{M}$, flux quantization must hold on the *entire family*, leading to:

$$\sum_i \int_{X_{4,i}} F_4 = \int_{\text{Fam}_\mathcal{M}} F_4 \in 2\pi\mathbb{Q} \quad (2.5)$$

and, by letting $X_{4,+}$ denote the compact space on one side of Σ and $X_{4,-}$ the right-hand compact space, we get

$$\int_{X_{4,+}} F_{4,+} - \int_{X_{4,-}} F_{4,-} = \int_\Sigma dC \quad (2.6)$$

This suggests that flux can discontinuously jump across the interstice, leading to a generalized conservation law where dC represents anomaly inflow at the interstice.

3 Bordism Correction Terms

The bordism group Ω_n^U contains torsion components, meaning that certain bordism classes do not bound a manifold but contribute discrete factors. This implies downstream constraints in, e.g., flux quantization.

Consider the *long exact sequence* in homology associated with a bordism W :

$$\dots \rightarrow H_{n+1}(W) \rightarrow H_n(\partial W) \rightarrow H_n(W) \rightarrow H_{n-1}(\partial W) \rightarrow \dots$$

Whence H_n has torsion, then we must modify the flux quantization condition by appending:

$$\int_{\Upsilon_{\text{out},1}} F_4 - \int_{\Upsilon_{\text{out},2}} = \int_{\Sigma} dB + \text{Tor}(\Omega_*^U) \quad (3.1)$$

where each of the $\Upsilon_{\text{out},*}$ are outputs of the bordism, $W : \Upsilon_{\text{in},0} \rightrightarrows \Upsilon_{\text{out},1} \times \Upsilon_{\text{out},2}$ is the bordism itself, and for every submanifold $\Upsilon_{\bullet,*}$, there is an associated Hilbert space $\mathcal{H}_{\Upsilon_{\bullet,*}}$ so that the bordism is a bifurcation of a unitary state living on $\mathcal{H}_{\Upsilon_{\text{in},0}}$. The output manifolds represent possible states⁴, which may be considered as entangled or in superposition.

Remark 3.1. *The Hilbert space splits as a pair of maps context slices over a spacetime groupoid:*

$$\mathcal{M}_{/\square} \xrightleftharpoons[g]{f} \mathcal{M}_{/\diamond_1} \times \mathcal{M}_{/\diamond_2}$$

where the collection of interstitial loci⁵ $[\wp] : \square$ may or may not be preserved, depending upon the severity of dissipation across time slices. We could, in principle, extend this to arbitrarily many future states by constructing a family

$$\text{Fam}_{\mathcal{M}_{/\diamond_{>0}}} = \text{Hom}(\mathcal{M}_{/\square}, \mathcal{M}_{/\diamond_{>0}})$$

and by letting

$$\square x = \diamond^0 x = \diamond_0$$

for all “packets” $x \in \mathcal{M}$.

To superize, we let $f \sim g$ for every function $(g \neq f) \in \text{Hom}(\mathcal{M}_{/\square}, \mathcal{M}_{/\diamond_{>0}})$ and take

$$\deg[f] = (-1)^{\deg \square + \deg \diamond_{>0}}$$

where $\deg$ is the Fredholm index. Then, if x is a quantum, its image under any $\tilde{f} \in [f]$ is a “squantum,” or superpartner.

3.1 Explicit Computations

The complex bordism groups Ω_n^U are given by the homotopy groups of the *Thom spectrum* MU :

$$\Omega_*^U = \pi_* MU$$

⁴Or, using the language of modal logic, $\diamond$ states

⁵Also known as “collar germs”

For low dimensions, the groups contain torsion elements; in particular:

$$\Omega_2^U = 0 \quad \Omega_4^U = \mathbb{Z} \quad \Omega_6^U = 0 \quad \Omega_8^U = \mathbb{Z}/2$$

The $\mathbb{Z}/2$ factor in Ω_8^U means that there exists an 8-manifold $\mathcal{M}^8$ which represents a non-trivial bordism class of order 2; that is, this manifold bounds a complex 9-manifold twice. Thus, we modify (3.1) appropriately:

$$\int_{\Sigma_{\text{interstice}}} dB \equiv 0 \pmod{2} \quad (3.2)$$

A well-known generator of the torsion in Ω_8^U is the complex projective space $\mathbb{C}P^4$ with a *specific framing*. The fundamental fact is that $2\mathbb{C}P^4$ is bordant to zero in Ω_8^U , meaning that twice this manifold bounds a 9-dimensional complex-oriented manifold.

Thus, we take

$$\mathcal{M}^8 \cong \mathbb{C}P^4$$

which carries a natural complex structure, with tangent bundle related to tautological line bundle $\mathcal{O}(-1)$ over $\mathbb{C}P^4$.

Since

$$H^4(\mathbb{C}P^4) = \mathbb{Z}$$

a flux F_4 on this space satisfies the quantization condition

$$\int_{\mathbb{C}P^4} F_4 \in \mathbb{Z}$$

However, if $\mathbb{C}P^4$ represents a torsion class in bordism, then twice this flux must trivialize in some 9-dimensional bounding space, which introduces a mod 2 quantization condition. So, we kick the can down the road a bit further, and modify (3.2) to become

$$\int_{\Upsilon_{\text{out},1}} F_4 - \int_{\Upsilon_{\text{out},2}} F_4 = \int_{\Sigma} dB + k \pmod{2} \quad (3.3)$$

thus constraining the spectrum of allowed flux values on the interstice.

3.2 Modified Partition Function

A quantum path integral must sum over all possible flux configuration, including the twisted sector. A good partition function would be:

$$\mathcal{Z} = \sum_{[F_4]} e^{iS[F_4]} \cdot e^{i\Theta([F_4])}$$

where $S[F_4]$ is the standard action term and $\Theta([F_4])$ is a torsion-dependent quantum phase factor that introduces interference between topologically distinct sectors.

A Pyknotic Objects

Recollection A.1. *A pyknotic structure means that objects carry a pro-finite topology, enabling a sort of inverse limit topology:*

$$\mathbf{A} \leftarrow \mathbf{A}_1 \leftarrow \dots \leftarrow \mathbf{A}_2 \dots$$

where each of the $\mathbf{A}_i$ are progressively higher resolution étale covers; $\mathbf{A}$ then emerges as the limiting object:

$$\mathbf{A} = \varprojlim \mathbf{A}_i$$

The limit $\varprojlim \mathbf{A}_i$ is naturally a stack over the étale site, which means that:

$$H_{\text{ét}}^n(\mathbf{A}) = \varprojlim H_{\text{ét}}^n(\mathbf{A}_i)$$

Remark A.1. *The above proof relies on the work of Barwick and Haine [8]. The condensation map $\mathbf{pyk}$ ensures that the category collapses to a smallest étale subcategory, effectively selecting the most refined structure. Since Zariski sheaves are already local in the étale topology, the condensation essentially restricts us to the “correct” structure.*

1-smallness follows from:

Proposition A.1. *The limit is well-defined in the étale topology, remains a 1-small subobject, and respects étale descent. Hence, condensation ensures that $\varprojlim \mathbf{A}_i$ stays fully faithful and remains in $\mathcal{Gpd}\mathcal{Ét}$.*

Proof. For any two objects x and y in the inverse system, the mapping space:

$$\text{Hom}_{\varprojlim \mathbf{A}_i}(x, y)$$

may be computed as a limit of the corresponding Hom-sets at each stage:

$$\varprojlim \text{Hom}_{\mathbf{A}_i}(x_i, y_i)$$

Since each $\mathbf{A}_i$ is étale, its morphisms are locally homeomorphic to a space over a base. That is, for every object x_i , there is a small neighborhood U_i such that $\pi : U_i \rightarrow S^\bullet$ is a local diffeomorphism. This means every morphism lifts uniquely in the étale topology, and there are no extra degrees of freedom in the inverse system.

The condensation property of $\mathbf{\sqsupset}$ ensures that at the limit, the étale transition maps glue uniquely; that is, there are no trivial kernels in our inverse limit. This is because pyknosis enforces Zariski-local triviality of the interstitial morphisms, meaning that all the lifting maps remain étale.

$\mathcal{Ét}\mathcal{Gpd}$ behaves as a single atomic object, and the proposition is shown to be true. $\square$

B Errata

B.1 Lemma 1.1

In the étale topology, smoothness must be established via local properties, such as the existence of atlases or effective descent in the context of Lie groupoids.

B.2 Higher Gauge Structures and Descent

Gauge transformations are usually local and glue over topological or étale covers via cocycle data.

Let $\{\lambda_i\}_{i \in I}$ be a collection of local gauge transformations defined over a cover $\{U_i\}$ of a base space X , with compatibility data λ_{ij} on overlaps $U_i \cap U_j$. A global gauge transformation Λ arises via descent if the stack of fields satisfies effective descent.

We avoid interpreting unless working in a category where pro-objects or formal completions are defined.

B.3 Bordism

The correct structure of the complex bordism group in dimension 8 is:

$$\Omega_8^U \cong \mathbb{Z} \oplus \mathbb{Z}/24$$

This replaces the earlier incorrect claim of $\mathbb{Z}/2\mathbb{Z}$. The presence of a free part and a 24-torsion component has implications for both cobordism classification and physical applications, such as anomaly cancellation and quantization.

Given the above correction, we replace

$$\int_{\Sigma} dB \cong 0 \pmod{2}$$

by

$$\int_{\Sigma} dB \cong 0 \pmod{24}$$

B.4 Relevance to SUGRA

Latly, we address concerns of relevance to 11-dimensional supergravity. The interstices are interpreted as codimension-1 topological defects or domain walls. Their interaction with the equations of motion in 11D supergravity must be made explicit.

The field equation is

$$d * G = \frac{1}{2} G \wedge G$$

and, to be physically consistent, the interstices should either

- source G , as in brane-like objects (e.g. $M9$ -branes), or
- mediate transitions between different flux vacua, like phase interfaces.

We propose modeling them via jump conditions or cobordism data between field configurations, though a detailed treatment is left for future work.

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D Disclaimer

Artificial Intelligence was used for the following purpose(s): Construction of Appendix A